

# The adjacency matroid of a graph

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## Abstract

If  $G$  is a looped graph, then its adjacency matrix represents a binary matroid  $M_A(G)$  on  $V(G)$ .  $M_A(G)$  may be obtained from the delta-matroid represented by the adjacency matrix of  $G$ , but  $M_A(G)$  is less sensitive to the structure of  $G$ . Jaeger proved that every binary matroid is  $M_A(G)$  for some  $G$  [*Ann. Discrete Math.* **17** (1983), 371-376].

The relationship between the matroidal structure of  $M_A(G)$  and the graphical structure of  $G$  has many interesting features. For instance, the matroid minors  $M_A(G) - v$  and  $M_A(G)/v$  are both of the form  $M_A(G' - v)$  where  $G'$  may be obtained from  $G$  using local complementation. In addition, matroidal considerations lead to a principal vertex tripartition, analogous in some ways to the principal edge tripartition of Rosenstiehl and Read [*Ann. Discrete Math.* **3** (1978), 195-226]. Several of these results are given two very different proofs, the first involving linear algebra and the second involving set systems or  $\Delta$ -matroids. Also, the Tutte polynomials of the adjacency matroids of  $G$  and its full subgraphs are closely connected to the interlace polynomial of Arratia, Bollobás and Sorkin [*Combinatorica* **24** (2004), 567-584].

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## 1 Introduction

A distinctive feature of matroid theory is that there are so many equivalent ways to define matroids, each providing its own special insight into the nature of the structure being defined. We refer to the books of Oxley [22], Welsh [28] and White [29, 30, 31] for thorough discussions. Here is one way to define a particular kind of matroid:

**Definition 1** A binary matroid  $M$  is an ordered pair  $(V, \mathcal{C}(M))$ , which satisfies the following circuit axioms:

1.  $V$  is a finite set and  $\mathcal{C}(M) \subseteq 2^V$ .
2.  $\emptyset \notin \mathcal{C}(M)$ .
3. If  $C_1, C_2 \in \mathcal{C}(M)$  then  $C_1 \not\subseteq C_2$ .
4. If  $C_1 \neq C_2 \in \mathcal{C}(M)$  then the symmetric difference  $(C_1 \setminus C_2) \cup (C_2 \setminus C_1) = C_1 \Delta C_2$  contains at least one  $C \in \mathcal{C}(M)$ .

If  $M$  and  $M'$  are matroids on  $V$  and  $V'$  then  $M \cong M'$  if there is a bijection between  $V$  and  $V'$  under which  $\mathcal{C}(M)$  and  $\mathcal{C}(M')$  correspond.

We consider  $2^V$  as a vector space over  $GF(2)$  in the usual way: if  $S, S_1, S_2 \subseteq V$  then  $0 \cdot S = \emptyset$ ,  $1 \cdot S = S$  and  $S_1 + S_2 = S_1 \Delta S_2$ .

**Definition 2** If  $M$  is a binary matroid on  $V$  then the cycle space  $Z(M)$  is the subspace of  $2^V$  spanned by  $\mathcal{C}(M)$ .

The importance of the cycle space of a binary matroid is reflected in the well known fact that two fundamental ideas of matroid theory, nullity and duality, correspond under  $Z$  to two fundamental ideas of linear algebra, dimension and orthogonality:  $\dim Z(M)$  is the nullity of  $M$ , and if  $M^*$  is the dual of  $M$  then  $Z(M^*)$  is the orthogonal complement of  $Z(M)$ . (See [22, 28, 30] for details.) Ghouila-Houri [17] showed that the importance of the cycle space is reflected in another special property, mentioned by some authors [7, 19, 30] but not stated explicitly in most accounts of the theory.

**Theorem 3** [17] Let  $V$  be a finite set. Then the function

$$Z : \{\text{binary matroids on } V\} \rightarrow \{GF(2)\text{-subspaces of } 2^V\}$$

is bijective.

Theorem 3 tells us that any construction or function which assigns a subspace of  $2^V$  to some object may be unambiguously reinterpreted as assigning a binary matroid to that object. There are of course many notions of linear algebra that involve assigning subspaces to objects. For instance, an  $m \times n$  matrix  $A$  with entries in  $GF(2)$  has four associated subspaces: the row space and right nullspace are orthogonal complements in  $GF(2)^n$ , and the column space and left nullspace are orthogonal complements in  $GF(2)^m$ . According to Theorem 3, we could just as easily say that an  $m \times n$  matrix with entries in  $GF(2)$  has four associated binary matroids, a pair of duals on an  $m$ -element set, and a pair of duals on an  $n$ -element set. For a symmetric matrix the row space and the column space are the same, and the left and right nullspace are the same.

Let  $G$  be a graph. A familiar construction associates to  $G$  its *polygon matroid*  $M(G)$ , the binary matroid on  $E(G)$  whose circuits are the minimal edge-sets of circuits of  $G$ . In this paper we discuss a different way to associate a binary matroid to  $G$ , which was mentioned by Jaeger in 1983 [19, 20]; the notion seems to have received little attention in the intervening decades.

**Definition 4** Let  $G$  be a graph, and let  $\mathcal{A}(G)$  be the Boolean adjacency matrix of  $G$ , i.e., the  $V(G) \times V(G)$  matrix with entries in  $GF(2)$ , in which a diagonal entry  $a_{vv}$  is 1 if and only if  $v$  is looped and an off-diagonal entry  $a_{vw}$  is 1 if and only if  $v \neq w$  are adjacent. Then the adjacency matroid  $M_A(G)$  is the binary matroid on  $V = V(G)$  represented by  $\mathcal{A}(G)$ , i.e., its circuits are the minimal nonempty subsets  $S \subseteq V$  such that the columns of  $\mathcal{A}(G)$  corresponding to elements of  $S$  are linearly dependent.

Here are four comments on Definition 4:

1. We understand the term *graph* to include multigraphs; that is, we allow graphs to have loops and parallel edges. Although Definition 4 applies to an arbitrary graph  $G$ ,  $\mathcal{A}(G)$  does not reflect the number of edges connecting two adjacent vertices, or the number of loops on a looped vertex. Consequently, the reader may prefer to think of  $\mathcal{A}(G)$  and  $M_A(G)$  as defined only when  $G$  is a looped simple graph.

2. In light of Theorem 3,  $M_A(G)$  may be described more simply as the binary matroid whose cycle space  $Z(M_A(G))$  is the nullspace of  $\mathcal{A}(G)$ .

3. The circuits of  $M_A(G)$  may also be described as the minimal nonempty subsets  $S \subseteq V(G)$  that satisfy a rather complicated neighbourhood parity condition: for every  $v \notin S$ ,  $|S \cap N(v)|$  is even; for every unlooped  $v \in S$ ,  $|S \cap N(v)|$  is even; and for every looped  $v \in S$ ,  $|S \cap N(v)|$  is odd. ( $N(v)$  denotes the set of vertices  $w \neq v$  that are adjacent to  $v$ .)

4. Many graph-theoretic properties of a graph  $G$  do not match conveniently with matroid-theoretic properties of  $M_A(G)$ . For example, recall that a *loop* in a matroid  $M$  is an element  $\lambda$  such that  $\{\lambda\} \in \mathcal{C}(M)$ , and a *coloop* is an element  $\kappa$  such that  $\kappa \notin \gamma$  for all  $\gamma \in \mathcal{C}(M)$ . In the polygon matroid of  $G$ , a loop is an edge incident on only one vertex, and a coloop is a cut edge. In the adjacency matroid of  $G$  we have the following very different results; (i) tells us that looped vertices of  $G$  cannot be loops of  $M_A(G)$ , and (ii) tells us that coloops of  $M_A(G)$  cannot in general have anything to do with connectedness of  $G$ . (Result (i) follows immediately from Definition 4, but proving (ii) requires a little more work; it follows readily from Lemma 27 below.)

(i) A vertex  $v \in V(G)$  is a loop of  $M_A(G)$  if and only if  $v$  is isolated and not looped in  $G$ .

(ii) Suppose  $v \in V(G)$ , and let  $G'$  be the graph obtained from  $G$  by toggling the loop status of  $v$ . Then  $v$  is a coloop of at least one of the adjacency matroids  $M_A(G)$ ,  $M_A(G')$ .

Theorem 3 and the equality  $Z(M^*) = Z(M)^\perp$  directly imply the following.

**Theorem 5** Let  $G$  and  $G'$  be two  $n$ -vertex graphs, let  $f : V(G) \rightarrow V(G')$  be a bijection, and let  $2^f : 2^{V(G)} \rightarrow 2^{V(G')}$  be the isomorphism of  $GF(2)$ -vector spaces induced by  $f$ . Then the following three conditions are equivalent.

1.  $f$  defines an isomorphism  $M_A(G) \cong M_A(G')$ .
2.  $2^f$  maps the column space of  $\mathcal{A}(G)$  onto the column space of  $\mathcal{A}(G')$ .
3.  $2^f$  maps the nullspace of  $\mathcal{A}(G)$  onto the nullspace of  $\mathcal{A}(G')$ .

Also, the following three conditions are equivalent.

1.  $f$  defines an isomorphism  $M_A(G) \cong M_A(G')^*$ .
2.  $2^f$  maps the column space of  $\mathcal{A}(G)$  onto the nullspace of  $\mathcal{A}(G')$ .
3.  $2^f$  maps the nullspace of  $\mathcal{A}(G)$  onto the column space of  $\mathcal{A}(G')$ .

Every graph with at least one edge has the same adjacency matroid as infinitely many other graphs, obtained by adjoining parallels. Even among looped simple graphs, there are many examples of nonisomorphic graphs with isomorphic adjacency matroids. For instance, the simple path of length two has the same adjacency matroid as the graph that consists of two isolated, looped vertices. However, a looped simple graph is determined up to isomorphism by the adjacency matroids of its full subgraphs.

**Definition 6** Let  $G$  be a graph, and suppose  $S \subseteq V(G)$ . Then  $G[S]$  denotes the full subgraph of  $G$  induced by  $S$ , i.e., the subgraph with  $V(G[S]) = S$  that includes the same incident edges as  $G$ .

If  $v \in V(G)$  then  $G[V(G) \setminus \{v\}]$  is also denoted  $G - v$ .

**Theorem 7** Let  $G$  and  $G'$  be looped simple graphs, and let  $f : V(G) \rightarrow V(G')$  be a bijection. Then the following are equivalent.

1.  $f$  is an isomorphism of graphs.
2. For every  $S \subseteq V(G)$ ,  $f$  defines an isomorphism  $M_A(G[S]) \cong M_A(G'[f(S)])$  of matroids.
3. For every  $S \subseteq V(G)$  with  $|S| \leq 2$ , the matroids  $M_A(G[S])$  and  $M_A(G'[f(S)])$  have the same nullity.

**Proof.** The implications  $1 \Rightarrow 2 \Rightarrow 3$  are obvious. The implication  $3 \Rightarrow 1$  follows from these facts: A vertex  $v \in V(G)$  is looped (resp. unlooped) if and only if the nullity of  $M_A(G[\{v\}])$  is 0 (resp. 1). If  $v \neq w \in V(G)$  are both unlooped, then they are adjacent (resp. nonadjacent) in  $G$  if and only if the nullity of  $M_A(G[\{v, w\}])$  is 0 (resp. 2). If  $v \in V(G)$  is looped and  $w \in V(G)$  is unlooped, then they are adjacent (resp. nonadjacent) in  $G$  if and only if the nullity of  $M_A(G[\{v, w\}])$  is 0 (resp. 1). If  $v \neq w \in V(G)$  are both looped, then they are adjacent (resp. nonadjacent) in  $G$  if and only if the nullity of  $M_A(G[\{v, w\}])$  is 1 (resp. 0). ■

The polygon matroids of graphs constitute a special subclass of the binary matroids. Jaeger [20] proved that the adjacency matroids of graphs, instead, include all the binary matroids:

**Theorem 8** [20] Let  $M$  be an arbitrary binary matroid. Then there is a graph  $G$  with  $M = M_A(G)$ .

Our purpose in writing this paper is to present the basic theory of the adjacency matroids of graphs. It turns out that this theory is closely connected to two important notions that have been studied by many researchers: the  $\Delta$ -matroids introduced by Bouchet [8, 9, 10] and the interlace polynomials introduced by Arratia, Bollobás and Sorkin [2, 3, 4]. These close connections are indicated by the fact that local complementation plays a significant role in all three theories.

**Definition 9** Let  $G$  be a graph with a vertex  $v$ . Then the local complement  $G^v$  is the looped simple graph obtained from  $G$  by toggling the loop status of every neighbour of  $v$ , and toggling the adjacency status of every pair of neighbours of  $v$ .

To be explicit:  $G^v$  is the looped simple graph related to  $G$  as follows: if  $v \neq w \neq x \neq v$  and both  $w$  and  $x$  are neighbours of  $v$  in  $G$ , then  $G^v$  has an edge  $wx$  if and only if  $w$  and  $x$  are not adjacent in  $G$ ; if  $w \neq x \in V(G)$  and at least one of  $w, x$  is not a neighbour of  $v$  in  $G$ , then  $G^v$  has an edge  $wx$  if and only if  $w$  and  $x$  are adjacent in  $G$ ; if  $w \neq v$  is a neighbour of  $v$  in  $G$  then there is a loop on  $w$  in  $G^v$  if and only if  $w$  is not looped in  $G$ ; and if  $w$  is not a neighbour of  $v$  in  $G$  then there is a loop on  $w$  in  $G^v$  if and only if  $w$  is looped in  $G$ . Note that for every graph  $G$ ,  $(G^v)^v$  is the looped simple graph obtained from  $G$  by replacing each set of parallels with a single edge.

There are two *matroid minor* operations, deletion and contraction.

**Definition 10** If  $M$  is a matroid on  $V$  and  $v \in V$  then the deletion  $M - v$  is the matroid on  $V \setminus \{v\}$  with  $\mathcal{C}(M - v) = \{\gamma \in \mathcal{C}(M) \mid v \notin \gamma\}$ .

If  $M$  is a binary matroid, then  $M - v$  is the binary matroid on  $V \setminus \{v\}$  with  $Z(M - v) = Z(M) \cap 2^{V \setminus \{v\}}$ .

**Definition 11** If  $M$  is a matroid on  $V$  and  $v \in V$ , then the contraction  $M/v$  is the matroid on  $V \setminus \{v\}$  with  $\mathcal{C}(M/v) = \{\text{minimal nonempty subsets } \gamma \subseteq V \setminus \{v\} \mid \gamma \cup \{v\} \text{ contains an element of } \mathcal{C}(M)\}$ .

If  $M$  is a binary matroid, let  $[v]$  denote the subspace of  $2^V$  spanned by  $\{v\}$ ; we identify  $2^{V \setminus \{v\}}$  with  $2^V/[v]$  in the natural way. Then  $M/v$  is the binary matroid on  $V \setminus \{v\}$  with  $Z(M/v) = (Z(M) + [v])/[v]$ .

Our first indication of the importance of local complementation for adjacency matroids is the fact that the matroid minors  $M_A(G)/v$  and  $M_A(G) - v$  can always be obtained by deleting  $v$  from graphs related to  $G$  through local complementation.

**Theorem 12** If  $v \in V(G)$  is a looped vertex then  $M_A(G)/v = M_A(G^v - v)$ .

**Theorem 13** Suppose  $v$  is an unlooped vertex of  $G$ .

1. If  $v$  is isolated then  $M_A(G)/v = M_A(G - v)$ .
2. If  $w$  is an unlooped neighbour of  $v$  then  $M_A(G)/v = M_A((G^w)^v - v)$ .
3. If  $w$  is a looped neighbour of  $G$  then  $M_A(G)/v = M_A(((G^v)^w)^v - v)$ .

**Theorem 14** If  $v$  is not a coloop of  $M_A(G)$  then  $M_A(G) - v = M_A(G - v)$ .

In general, Theorem 14 fails for coloops. For example, let  $v$  and  $w$  be the vertices of the simple path  $P_2$  of length two. Then  $w$  is isolated and unlooped in  $P_2 - v$ ; consequently  $\mathcal{C}(M_A(P_2 - v)) = \{\{w\}\}$  even though  $\mathcal{C}(M_A(P_2)) = \emptyset$ .

Observe that if  $M$  is a matroid on  $V$  and  $v \in V$ , then  $M - v = M/v$  if and only if  $v$  is either a loop or a coloop. (For if  $v$  appears in any circuit  $\gamma$  with  $|\gamma| > 1$  then  $\gamma \setminus \{v\}$  contains a circuit of  $M/v$  but  $\gamma \setminus \{v\}$  contains no circuit of  $M - v$ .) It follows that the failure of Theorem 14 for coloops is not a significant inconvenience: if  $v$  is a coloop of  $M_A(G)$  then we may refer to Theorem 12 or Theorem 13 to describe  $M_A(G) - v = M_A(G)/v$ .

Another instance of the importance of local complementation is the fact that matroid deletions from  $M_A(G)$  and  $M_A(G^v)$  always coincide.

**Theorem 15** *If  $v \in V(G)$  then  $M_A(G) - v = M_A(G^v) - v$ .*

The connection between the entire matroids  $M_A(G)$  and  $M_A(G^v)$  is more complicated. Recall that if  $M$  and  $M'$  are matroids on disjoint sets  $V$  and  $V'$  then their *direct sum*  $M \oplus M'$  is the matroid on  $V \cup V'$  with  $\mathcal{C}(M \oplus M') = \mathcal{C}(M) \cup \mathcal{C}(M')$ . Also, if  $v$  is a single element then  $U_{1,1}(\{v\})$  denotes the matroid on  $\{v\}$  in which  $v$  is a coloop, and  $U_{1,0}(\{v\})$  denotes the matroid on  $\{v\}$  in which  $v$  is a loop. Clearly  $M = (M - v) \oplus U_{1,1}(\{v\})$  if and only if  $v$  is a coloop of  $M$ , and  $M = (M - v) \oplus U_{1,0}(\{v\})$  if and only if  $v$  is a loop of  $M$ .

**Theorem 16** *1. If  $v \in V(G)$  is unlooped then  $M_A(G^v) = M_A(G)$ .*

*2. If  $v \in V(G)$  is a coloop of both  $M_A(G)$  and  $M_A(G^v)$ , then  $M_A(G^v) = M_A(G)$ .*

*3. If  $v \in V(G)$  is looped and not a coloop of one of  $M_A(G), M_A(G^v)$ , then  $v$  is a coloop of the other and  $M_A(G^v) \not\cong M_A(G)$ . More specifically, if  $\{M_A(G), M_A(G^v)\} = \{M_1, M_2\}$  with  $v$  not a coloop of  $M_1$ , then  $v$  is a coloop of  $M_2$  and  $M_2 = (M_1 - v) \oplus U_{1,1}(\{v\})$ .*

Unlike Theorem 5, Theorem 16 does not explain all isomorphisms of adjacency matroids. For instance, the simple path of length two has the same adjacency matroid as a disconnected graph consisting of two looped vertices; but local complementation cannot disconnect a connected graph.

Theorems 12 – 16 are proven in Section 3, using elementary linear algebra. In Section 4 we discuss Theorem 18, which provides another illustration of the connections tying adjacency matroids to delta-matroids and the theory of the interlace polynomials. It is a matroid version of Lemma 2 of Balister, Bollobás, Cutler, and Pebody [5]. A preliminary definition will be useful.

**Definition 17** *If  $v$  is a vertex of  $G$  then  $G(v)$  denotes the graph obtained from  $G$  by removing every loop incident on  $v$ ,  $G(v, \ell)$  denotes the graph obtained from  $G(v)$  by attaching a loop at  $v$ , and  $G(v, li)$  denotes the graph obtained from  $G(v, \ell)$  by isolating  $v$  (i.e., removing all non-loop edges incident on  $v$ ).*

**Theorem 18** *Let  $v$  be a vertex of  $G$ . Then two of the three adjacency matroids  $M_A(G(v))$ ,  $M_A(G(v, \ell))$ ,  $M_A(G(v, li))$  are the same, and the other is different. The cycle space of the different matroid contains the cycle space shared by the two that are the same, and its dimension is greater by 1.*

The adjacency matroid  $M_A(G(v, \ell i))$  may be described in two other ways. As  $v$  is an isolated, looped vertex of  $G(v, \ell i)$ , it is a coloop of  $M_A(G(v, \ell i))$ ; hence  $M_A(G(v, \ell i)) = M_A(G(v, \ell i) - v) \oplus U_{1,1}(\{v\}) = M_A(G - v) \oplus U_{1,1}(\{v\})$ . Another description comes from Theorem 12, which tells us that  $M_A(G - v) = M_A(G^v(v, \ell))/v$ . Also, the fact that  $v$  is not a loop of  $M_A(G^v(v, \ell))$  implies that  $M_A(G^v(v, \ell))/v$  and  $M_A(G^v(v, \ell))$  have the same nullity. For ease of reference we state these observations as a proposition.

**Proposition 19** *If  $v \in V(G)$  then  $v$  is a coloop of*

$$M_A(G(v, \ell i)) = M_A(G - v) \oplus U_{1,1}(\{v\}) = (M_A(G^v(v, \ell))/v) \oplus U_{1,1}(\{v\}).$$

*Consequently  $M_A(G(v, \ell i))$ ,  $M_A(G - v)$ ,  $M_A(G^v(v, \ell))/v$  and  $M_A(G^v(v, \ell))$  all have the same nullity.*

Recall that according to property (ii) above,  $v$  must be a coloop of at least one of the matroids  $M_A(G(v))$ ,  $M_A(G(v, \ell))$ . Consequently  $v$  must fall under one (and only one) of these cases:

1.  $v$  is a coloop of both  $M_A(G(v))$  and  $M_A(G(v, \ell))$
2.  $v$  is a coloop of  $M_A(G(v))$  and not  $M_A(G(v, \ell))$
3.  $v$  is a coloop of  $M_A(G(v, \ell))$  and not  $M_A(G(v))$

As each vertex of  $G$  must fall under precisely one of the cases 1 – 3, we obtain a partition of  $V(G)$  into three subsets. We refer to this partition of  $V(G)$  as the *principal vertex tripartition* of  $G$ . In Section 5 we prove that the three subsets of the principal vertex tripartition correspond precisely to the three alternatives of Theorem 18:  $v$  falls under case 1 if and only if  $M_A(G(v)) = M_A(G(v, \ell))$ ;  $v$  falls under case 2 if and only if  $M_A(G(v)) = M_A(G(v, \ell i))$ ; and  $v$  falls under case 3 if and only if  $M_A(G(v, \ell)) = M_A(G(v, \ell i))$ . Moreover,  $v$  falls under case 1 in  $G$  if and only if  $v$  falls under case 2 in  $G^v$ , and vice versa. Theorem 20 below includes all of these results, and also gives a few more details; in particular, the final assertion of case 1 corrects an error in [24].

**Theorem 20** *Let  $v$  be a vertex of  $G$ . Then the list  $M_A(G(v))$ ,  $M_A(G(v, \ell))$ ,  $M_A(G(v, \ell i))$ ,  $M_A(G^v(v))$ ,  $M_A(G^v(v, \ell))$ ,  $M_A(G^v(v, \ell i))$  includes either two or three distinct matroids. Only one of these distinct matroids does not include  $v$  as a coloop, and this matroid determines the others as follows.*

1. *If  $v$  is a coloop of both  $M_A(G(v))$  and  $M_A(G(v, \ell))$  then  $v$  is not a coloop of  $M_A(G^v(v, \ell))$ ,*

$$\begin{aligned} M_A(G(v, \ell i)) &= (M_A(G^v(v, \ell))/v) \oplus U_{1,1}(\{v\}) \text{ and} \\ M_A(G(v)) &= M_A(G(v, \ell)) = M_A(G^v(v)) = M_A(G^v(v, \ell i)) \\ &= (M_A(G^v(v, \ell)) - v) \oplus U_{1,1}(\{v\}). \end{aligned}$$

$M_A(G(v, \ell_i))$  and  $M_A(G^v(v, \ell))$  have the same nullity, say  $\nu + 1$ ; the nullity of  $M_A(G(v))$  is  $\nu$  and

$$Z(M_A(G(v, \ell_i))) \cap Z(M_A(G^v(v, \ell))) = Z(M_A(G(v))).$$

This case requires that  $G^v - v$  have at least one looped vertex.

2. If  $v$  is not a coloop of  $M_A(G(v, \ell))$  then the assertions of case 1 hold, with the roles of  $G$  and  $G^v$  interchanged.

3. If  $v$  is not a coloop of  $M_A(G(v))$  then

$$\begin{aligned} M_A(G^v(v)) &= M_A(G(v)) \text{ and} \\ M_A(G(v, \ell)) &= M_A(G^v(v, \ell)) = M_A(G(v, \ell_i)) = M_A(G^v(v, \ell_i)) \\ &= (M_A(G(v)) - v) \oplus U_{1,1}(\{v\}). \end{aligned}$$

The nullity of  $M_A(G(v))$  is 1 more than the nullity of  $M_A(G(v, \ell))$ , and

$$Z(M_A(G(v, \ell))) \subset Z(M_A(G(v))).$$

The principal vertex tripartition is reminiscent of the principal edge tripartition of Rosenstiehl and Read [23], but there is a fundamental difference between the two tripartitions. The principal edge tripartition of  $G$  is determined by the polygon matroid of  $G$ , but the principal vertex tripartition of  $G$  is not determined by the adjacency matroid of  $G$ :

**Theorem 21** *The adjacency matroid and the principal vertex tripartition are independent, in the sense that two graphs may have isomorphic adjacency matroids and distinct principal vertex tripartitions, or nonisomorphic adjacency matroids and equivalent principal vertex tripartitions.*

After verifying Theorem 21 in Section 6, in Sections 7 – 9 we turn our attention to the close connection between adjacency matroids and  $\Delta$ -matroids.

**Definition 22** [8] *A delta-matroid ( $\Delta$ -matroid for short) is an ordered pair  $D = (V, \sigma)$  consisting of a finite set  $V$  and a nonempty family  $\sigma \subseteq 2^V$  that satisfies the symmetric exchange axiom: For all  $X, Y \in \sigma$  and all  $u \in X \Delta Y$ ,  $X \Delta \{u\} \in \sigma$  or there is a  $v \neq u \in X \Delta Y$  such that  $X \Delta \{u, v\} \in \sigma$  (or both).*

We often write  $X \in D$  rather than  $X \in \sigma$ . The name  $\Delta$ -matroid reflects the fact that if  $M$  is a matroid, the family of bases  $\mathcal{B}(M)$  satisfies the symmetric exchange property; indeed  $\mathcal{B}(M)$  satisfies the stronger *basis exchange axiom*: if  $X, Y \in \mathcal{B}(M)$  and  $u \in X \setminus Y$  then there is some  $v \in Y \setminus X$  with  $X \Delta \{u, v\} \in \mathcal{B}(M)$ .

**Definition 23** [8] *If  $G$  is a graph then its associated  $\Delta$ -matroid is  $\mathcal{D}_G = (V(G), \sigma)$  with*

$$\sigma = \{S \subseteq V(G) \mid \mathcal{A}(G[S]) \text{ is nonsingular}\}.$$



$\mathcal{D}_G$  is determined by the adjacency matroids  $M_A(G[S])$ :  $S \in \mathcal{D}_G$  if and only if  $M_A(G[S])$  is a free matroid (i.e.,  $\mathcal{C}(M_A(G[S])) = \emptyset$ ). There is more to the relationship between  $\mathcal{D}_G$  and the matroids  $M_A(G[S])$  than this obvious observation, though. Recall that if  $M$  is a matroid on  $V$  then an *independent set* of  $M$  is a subset of  $V$  that contains no circuit of  $M$ , and a *basis* of  $M$  is a maximal independent set. For  $S \subseteq V(G)$  let  $\mathcal{I}(M_A(G[S]))$  and  $\mathcal{B}(M_A(G[S]))$  denote the families of independent sets and bases (respectively) of the adjacency matroid  $M_A(G[S])$ .

**Theorem 24** *Let  $G$  be a graph, and suppose  $S \subseteq V(G)$ .*

1.  $M_A(G[S])$  is the matroid on  $S$  with

$$\mathcal{B}(M_A(G[S])) = \{\text{maximal } B \subseteq S \mid B \in \mathcal{D}_G\}.$$

2.  $M_A(G[S])$  is the matroid on  $S$  with

$$\mathcal{I}(M_A(G[S])) = \{I \subseteq S \mid \text{there is some } X \in \mathcal{D}_G \text{ with } I \subseteq X \subseteq S\}.$$

3.  $\mathcal{D}_{G[S]}$  is the  $\Delta$ -matroid  $(S, \sigma)$  with

$$\sigma = \bigcup_{T \subseteq S} \mathcal{B}(M_A(G[T])).$$

Although Theorem 24 applies to arbitrary subsets  $S \subseteq V(G)$ , the heart of the theorem is the result that part 1 holds for  $S = V(G)$ ; as noted by Brijder and Hoogeboom [12], this result is a special case of the strong principal minor theorem of Kodiyalam, Lam and Swan [21].

In Sections 8 and 9 we reprove Theorems 12 – 16 within the contexts of set systems and  $\Delta$ -matroids. In particular, Theorem 12 and Theorem 14 are generalized to set systems and  $\Delta$ -matroids, respectively. In a similar vein, Theorem 14 of [12] shows that some aspects of the principal vertex tripartition extend to matroids associated with arbitrary  $\Delta$ -matroids.

In Section 10 we discuss the connection between the interlace polynomials introduced by Arratia, Bollobás and Sorkin [2, 3, 4] and the Tutte polynomials of the adjacency matroids of a graph and its full subgraphs. This connection seems to be fundamentally different from the connection between the one-variable interlace polynomial of a planar circle graph and the Tutte polynomial of an associated checkerboard graph, discussed by Arratia, Bollobás and Sorkin [3] and Ellis-Monaghan and Sarmiento [16].

## 2 Theorems 3 and 8

For the convenience of the reader, in this section we provide proofs of theorems of Ghouila-Houri [17] and Jaeger [20] mentioned in the introduction.

It is well known that axiom 4 of Definition 1 may be replaced by the following seemingly stronger requirement [22, 28, 30]:

4'. If  $C_1, C_2, \dots, C_k \in \mathcal{C}(M)$  do not sum to  $\emptyset$  in  $2^V$  then there are pairwise disjoint  $C'_1, \dots, C'_{k'} \in \mathcal{C}(M)$  such that

$$\sum_{i=1}^k C_i = \bigcup_{i=1}^{k'} C'_i.$$

This axiom is useful in the proof of Theorem 3:

Let  $M$  be a binary matroid on  $V$ . We can certainly construct  $Z(M)$  from  $\mathcal{C}(M)$ , using the addition of  $2^V$ . It turns out that we can also construct  $\mathcal{C}(M)$  from  $Z(M)$ :

$$\mathcal{C}(M) = \{\text{minimal nonempty subsets of } V \text{ that appear in } Z(M)\}.$$

The proof is simple: axiom 4' implies that every nonempty element of  $Z(M)$  contains a circuit, so every minimal nonempty element of  $Z(M)$  is an element of  $\mathcal{C}(M)$ ; conversely, axiom 3 tells us that no circuit contains another, so it is impossible for a circuit to contain a minimal nonempty element of  $Z(M)$  other than itself.

This implies that the function  $Z$  is injective.

Now, let  $W$  be any subspace of  $2^V$ . If  $W = \{\emptyset\}$  then  $W = Z(U)$ , where  $U$  is the free matroid on  $V$  (i.e.,  $\mathcal{C}(U) = \emptyset$ ). Suppose  $\dim W \geq 1$ , and let  $\mathcal{C}(W)$  be the set of minimal nonempty subsets of  $V$  that appear in  $W$ . It is a simple matter to verify that  $\mathcal{C}(W)$  satisfies Definition 1; hence  $\mathcal{C}(W)$  is the circuit-set of a binary matroid  $M(W)$ . As  $\mathcal{C}(M(W)) = \mathcal{C}(W) \subseteq W$ , and  $Z(M(W))$  is spanned by  $\mathcal{C}(M(W))$ ,  $Z(M(W))$  is a subspace of  $W$ .

Could  $Z(M(W))$  be a proper subspace of  $W$ ? If so, then there is some  $w \in W$  that is not an element of  $Z(M(W))$ . By definition,  $\mathcal{C}(W)$  must include some element  $\gamma$  that is a subset of  $w$ . Then  $w + \gamma = w \Delta \gamma = w \setminus \gamma$  is also an element of  $W - Z(M(W))$ , and its cardinality is strictly less than that of  $w$ . We deduce that  $W - Z(M(W))$  does not have an element of smallest cardinality. This is ridiculous, so  $Z(M(W))$  cannot be a proper subspace of  $W$ .

It follows that the function  $Z$  is surjective. ■

In light of Theorems 3 and 5 of Section 1, proving Theorem 8 is the same as proving the following.

**Proposition 25** *Let  $A$  be a  $k \times n$  matrix with entries in  $GF(2)$ . Then there is a symmetric  $n \times n$  matrix  $B$  whose nullspace is the same as the right nullspace of  $A$ .*

**Proof.** If the right nullspace space of  $A$  is  $GF(2)^n$ , the proposition is satisfied by the zero matrix; if the right nullspace of  $A$  is  $\{\mathbf{0}\}$  then the proposition is satisfied by the identity matrix.

Otherwise, the right nullspace of  $A$  is a proper subspace of  $GF(2)^n$ . Using elementary row operations, we obtain from  $A$  an  $r \times n$  matrix  $C$  in echelon form, which has the same right nullspace as  $A$ . (Here  $r$  is the rank of  $A$ .) There is

a permutation  $\pi$  of  $\{1, \dots, n\}$  such that the matrix obtained by permuting the columns of  $C$  according to  $\pi$  is of the form

$$C' = (I_r \quad C'')$$

where  $I_r$  is an identity matrix. If  $(C'')^{tr}$  denotes the transpose of  $C''$ , then

$$B' = \begin{pmatrix} I_r & C'' \\ (C'')^{tr} & (C'')^{tr} \cdot C'' \end{pmatrix}$$

is a symmetric matrix.  $B'$  has the same right nullspace as  $C'$ , for if  $(I_r \quad C'') \cdot \kappa = \mathbf{0}$  then certainly

$$((C'')^{tr} \quad (C'')^{tr} \cdot C'') \cdot \kappa = (C'')^{tr} \cdot (I_r \quad C'') \cdot \kappa = \mathbf{0}.$$

Consequently, a matrix  $B$  satisfying the statement may be obtained by permuting the rows and columns of  $B'$  according to  $\pi^{-1}$ . ■

### 3 Local complementation and matroid minors

The reader familiar with the interlace polynomials of Arratia, Bollobás and Sorkin [2, 3, 4] will recognize some of the concepts and notation that appear in our discussion of adjacency matroids, but it is important to keep a significant difference in mind: The interlace polynomials of  $G$  are related to principal submatrices of  $\mathcal{A}(G)$ , i.e., square submatrices obtained from  $\mathcal{A}(G)$  by removing some columns and the corresponding rows. The adjacency matroid of  $G$ , instead, is related to rectangular submatrices obtained by removing only columns from  $\mathcal{A}(G)$ .

#### 3.1 Theorems 12 and 15

Suppose  $v \in V(G)$ ; let

$$\mathcal{A}(G) = \begin{pmatrix} * & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix} \text{ and } \mathcal{A}(G^v) = \begin{pmatrix} * & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix}.$$

Here bold numerals indicate rows and columns with all entries equal, the first row and column correspond to  $v$ , and  $A^c$  is the matrix obtained by toggling all the entries of  $A$ . To prove Theorem 15, observe that elementary row operations transform

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A & B \\ C & D \end{pmatrix} \text{ into } \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A^c & B \\ C & D \end{pmatrix}.$$

It follows that if  $\kappa$  is a column vector then

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A & B \\ C & D \end{pmatrix} \cdot \kappa = \mathbf{0} \text{ if and only if } \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A^c & B \\ C & D \end{pmatrix} \cdot \kappa = \mathbf{0}.$$

That is, these two matrices have the same right nullspace. It follows that  $Z(M_A(G) - v) = Z(M_A(G^v) - v)$ , and hence  $M_A(G) - v = M_A(G^v) - v$ , as asserted by Theorem 15.

Theorem 12 follows from another calculation using elementary row operations. Matrices of the forms

$$\begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix} \text{ and } \begin{pmatrix} A^c & B \\ C & D \end{pmatrix}$$

have the same  $GF(2)$ -nullity, so if  $v$  is looped and  $v \notin S \subseteq V(G)$  then  $S \cup \{v\}$  contains a circuit of  $M_A(G)$  if and only if  $S$  contains a circuit of  $M_A(G^v - v)$ . It follows that if  $v$  is looped, then  $M_A(G)/v = M_A(G^v - v)$ .

### 3.2 Theorem 14 and the strong principal minor theorem

We turn now to Theorem 14. Suppose  $v$  is not a coloop of  $M_A(G)$ . Then  $\mathcal{A}(G)$  is a symmetric matrix of the form

$$\begin{pmatrix} * & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix},$$

with the first column corresponding to  $v$  and equal to the sum of certain other columns. It follows that

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

are related through elementary row operations, so these two matrices have the same right nullspace:  $Z(M_A(G) - v) = Z(M_A(G - v))$ . Consequently  $M_A(G) - v = M_A(G - v)$ , as asserted by Theorem 14.

By the way, note that this argument still applies if  $v$  is a coloop, provided that  $v$  is not a coloop of the adjacency matroid of the graph obtained from  $G$  by toggling the loop status of  $v$ .

Theorem 14 is equivalent to the following special case of the strong principal minor theorem of Kodiyalam, Lam and Swan [21].

**Theorem 26** *Let  $A$  be a symmetric  $n \times n$  matrix with entries in  $GF(2)$  and let  $S$  be a subset of  $\{1, \dots, n\}$ , of size  $r = \text{rank}(A)$ . Then the columns of  $A$  corresponding to elements of  $S$  are linearly independent if and only if the principal submatrix of  $A$  corresponding to  $S$  is nonsingular.*

**Proof.** If the principal submatrix of  $A$  corresponding to  $S$  is nonsingular then its columns must be linearly independent. Obviously then the corresponding columns of  $A$ , which are obtained from the columns of the principal submatrix by inserting rows corresponding to elements of  $\{1, \dots, n\} \setminus S$ , must also be linearly independent.

The interesting part of the theorem is the converse: if the columns of  $A$  corresponding to elements of  $S$  form an  $n \times r$  matrix of rank  $r$ , then the  $r \times r$  submatrix obtained by removing the rows corresponding to elements of  $\{1, \dots, n\} \setminus S$  is also of rank  $r$ . The proof is simple: Let  $A'$  be the  $n \times r$  submatrix of  $A$  that includes only the columns with indices from  $S$ ; by hypothesis,  $A$  and  $A'$  have the same column space. If  $i \in \{1, \dots, n\} \setminus S$  then the  $i$ th column of  $A$  must be the sum of some columns with indices from  $S$ , and by symmetry the  $i$ th row of  $A$  must be the sum of some rows with indices from  $S$ . Consequently the same is true of the  $i$ th row of  $A'$ . It follows that removing the  $i$ th row of  $A'$  for every  $i \notin S$  yields an  $r \times r$  submatrix with the same row space as  $A'$ , and hence the same rank as  $A$ . ■

### 3.3 Theorems 13 and 16

To prove part 1 of Theorem 16, suppose  $v \in V(G)$  is unlooped. Let

$$\mathcal{A}(G) = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix} \text{ and } \mathcal{A}(G^v) = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix}.$$

Elementary row operations transform  $\mathcal{A}(G)$  into  $\mathcal{A}(G^c)$ , so these matrices have the same right nullspace. It follows that  $Z(M_A(G)) = Z(M_A(G^v))$ .

We are now ready to prove all three parts of Theorem 13. If  $v$  is unlooped and isolated then  $v$  is a loop in  $M_A(G)$ , so  $M_A(G)/v = M_A(G) - v$ ; Theorem 14 tells us that  $M_A(G) - v = M_A(G - v)$ . If  $v$  and  $w$  are unlooped neighbours then part 1 of Theorem 16 tells us that  $M_A(G) = M_A(G^w)$ ;  $v$  is looped in  $G^w$ , so Theorem 12 tells us that  $M_A(G^w)/v = M_A((G^w)^v - v)$ . Finally, if  $v$  is unlooped and  $w$  is a looped neighbour of  $v$  in  $G$  then part 1 of Theorem 16 tells us that  $M_A(G) = M_A(G^v)$ ;  $w$  is an unlooped neighbour of  $v$  in  $G^v$ , so the preceding sentence tells us that  $M_A(G)/v = M_A(G^v)/v = M_A(((G^v)^w)^v - v)$ .

Turning to part 2 of Theorem 16, suppose a looped vertex  $v$  is a coloop of both  $M_A(G)$  and  $M_A(G^v)$ . Then  $\mathcal{C}(M_A(G)) = \mathcal{C}(M_A(G) - v)$  and  $\mathcal{C}(M_A(G^v)) = \mathcal{C}(M_A(G^v) - v)$ . Theorem 15 tells us that  $M_A(G) - v = M_A(G^v) - v$ , so  $M_A(G) = M_A(G^v)$ .

Part 3 of Theorem 16 involves the following.

**Lemma 27** *Suppose  $v \in V(G)$ . Then  $v$  is not a coloop of  $M_A(G)$  if and only if the three matrices*

$$\mathcal{A}(G) = \begin{pmatrix} * & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix}, \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ A & B \\ C & D \end{pmatrix}$$

*have the same rank over  $GF(2)$ . (Here the first row and column of  $\mathcal{A}(G)$  correspond to  $v$ .)*

**Proof.** If the three matrices have the same rank then in particular, the first two have the same rank. Consequently the first column of  $\mathcal{A}(G)$  must

equal the sum of certain other columns; hence there is a circuit of  $M_A(G)$  that contains  $v$ . Conversely, if  $v$  is not a coloop of  $M_A(G)$  then the column of  $\mathcal{A}(G)$  corresponding to  $v$  must be the sum of the columns corresponding to some subset  $S_v \subseteq V(G) \setminus \{v\}$ . By symmetry, the sum of the rows corresponding to  $S_v$  must equal the first row of  $\mathcal{A}(G)$ . ■

Suppose now that  $v$  is a looped non-coloop of  $M_A(G)$ . The set  $S_v$  must include an odd number of columns of  $A$ , to yield the diagonal entry  $* = 1$  in the column of  $\mathcal{A}(G)$  corresponding to  $v$ . Replacing  $A$  with  $A^c$  toggles an odd number of summands in each row of  $A$ , so the sum of the columns of

$$\mathcal{A}(G^v) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix}$$

corresponding to  $S_v$  must be the column vector

$$\begin{pmatrix} 1 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

It follows that the  $GF(2)$ -ranks of

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ A^c & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix}$$

are the same. According to Lemma 27,  $v$  cannot be a non-coloop of  $M_A(G^v)$ .

By the way, the same argument shows that removing the loop from  $v$  cannot produce a non-coloop in the adjacency matroid of the resulting graph. That is, in the terminology of Section 4  $v$  is a triple coloop of  $M_A(G^v)$ .

To complete the proof of part 3 of Theorem 16, note that the fact that  $v$  is a coloop of  $M_A(G^v)$  implies that  $\mathcal{C}(M_A(G^v)) = \mathcal{C}(M_A(G^v) - v)$ . Theorem 15 tells us that  $\mathcal{C}(M_A(G^v) - v) = \mathcal{C}(M_A(G) - v)$ , and the fact that  $v$  is not a coloop of  $M_A(G)$  implies that  $\mathcal{C}(M_A(G) - v)$  is a proper subset of  $\mathcal{C}(M_A(G))$ . It follows that  $|\mathcal{C}(M_A(G^v))| < |\mathcal{C}(M_A(G))|$ , and consequently  $M_A(G^v) \not\cong M_A(G)$ . The equality  $M_A(G^v) = (M_A(G^v) - v) \oplus U_{1,1}(\{v\})$  follows immediately from the fact that  $v$  is a coloop of  $M_A(G^v)$ , and this equality implies  $M_A(G^v) = (M_A(G) - v) \oplus U_{1,1}(\{v\})$  by Theorem 15.

## 4 Theorem 18 and triple coloops

Theorem 18 is essentially a result about the nullspaces of  $\mathcal{A}(G(v))$ ,  $\mathcal{A}(G(v, \ell))$  and  $\mathcal{A}(G(v, \ell i))$ . With a convenient order on the vertices of  $G$ , these three matrices are

$$\begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix}, \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & B \\ \mathbf{0} & C & D \end{pmatrix}$$

respectively. Theorem 18 asserts that two of the nullspaces are the same, say of dimension  $\nu$ ; the different nullspace contains them, and its dimension is  $\nu + 1$ . A proof of this statement is given in [25].

Observe that no element of the nullspace of the right-hand matrix could possibly have a nonzero first coordinate. Consequently  $v$  does not appear in any circuit of  $M_A(G(v, \ell i))$ ; that is,  $v$  is a coloop of  $M_A(G(v, \ell i))$ , as noted in Proposition 19 of the introduction. In the special case that  $M_A(G(v, \ell i))$  has a larger cycle space than  $M_A(G(v)) = M_A(G(v, \ell))$ , it follows that  $v$  must also be a coloop of  $M_A(G(v))$  and  $M_A(G(v, \ell))$ . On the other hand, if  $M_A(G(v))$  or  $M_A(G(v, \ell))$  has a larger cycle space than  $M_A(G(v, \ell i))$  then either the matrix displayed on the left or the matrix displayed in the center has a larger nullspace than the one on the right. Clearly any vector in either of these two nullspaces that is not in the nullspace of the right-hand matrix must have a nonzero first coordinate; consequently  $v$  is not a coloop of the corresponding matroid. We deduce the following sharpened form of the result (ii) mentioned in the introduction.

**Corollary 28** *If  $v \in V(G)$  then  $v$  is a coloop of  $M_A(G(v, \ell i))$  and at least one of the adjacency matroids  $M_A(G(v))$ ,  $M_A(G(v, \ell))$ . It is a coloop of all three if and only if*

$$Z(M_A(G(v))) = Z(M_A(G(v, \ell))) \subset Z(M_A(G(v, \ell i))).$$

The special case in which  $v$  is a coloop of all three matroids is important enough to merit a special name.

**Definition 29** *A vertex  $v \in V(G)$  is a triple coloop of  $M_A(G)$  if it is a coloop of  $M_A(G(v))$ ,  $M_A(G(v, \ell))$ , and  $M_A(G(v, \ell i))$ .*

Note that the nomenclature is imprecise; although a triple coloop of  $M_A(G)$  is certainly a coloop of  $M_A(G)$ , it is not the matroid structure of  $M_A(G)$  that determines whether or not a vertex is a triple coloop. We prefer this imprecise nomenclature over the alternative “ $v$  is a triple coloop of  $G$ ” because that would also be confusing; a coloop (isthmus) of  $G$  is an edge, not a vertex.

Using this notion, Theorems 14 and 16 may be sharpened as follows.

**Theorem 30** 1. *If  $v \in V(G)$  is unlooped then  $M_A(G^v) = M_A(G)$ .*

2. *If a looped vertex  $v \in V(G)$  is a coloop of both  $M_A(G)$  and  $M_A(G^v)$ , then  $M_A(G^v) = M_A(G)$  and  $v$  is not a triple coloop of  $M_A(G^v)$  or  $M_A(G)$ .*

3. *If  $v \in V(G)$  is looped and not a coloop of one of  $M_A(G)$ ,  $M_A(G^v)$ , then  $v$  is a triple coloop of the other and  $M_A(G^v) \not\cong M_A(G)$ . More specifically, if  $\{M_A(G), M_A(G^v)\} = \{M_1, M_2\}$  with  $v$  not a coloop of  $M_1$ , then  $v$  is a triple coloop of  $M_2$  and  $M_2 = (M_1 - v) \oplus U_{1,1}(\{v\})$ .*

**Theorem 31** *If  $v \in V(G)$  is not a triple coloop of  $M_A(G)$ , then  $M_A(G) - v = M_A(G - v)$ .*

**Proof.** Part 1 of Theorem 30 is the same as part 1 of Theorem 16. The proofs of Theorem 31 and part 3 of Theorem 30 are indicated in the preceding section; both are introduced with the phrase “by the way.”

It remains to consider part 2 of Theorem 30. Suppose a looped vertex  $v \in V(G)$  is a triple coloop of  $G$ , and let

$$\mathcal{A}(G) = \mathcal{A}(G(v, \ell)) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix}.$$

According to Theorem 18 and Corollary 28,

$$\nu(\mathcal{A}(G(v))) = \nu(\mathcal{A}(G(v, \ell))) = \nu(\mathcal{A}(G(v, \ell i))) - 1,$$

where  $\nu$  denotes the  $GF(2)$ -nullity. Observe that elementary row and column operations transform  $\mathcal{A}(G(v, \ell))$  into

$$\begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix},$$

which is  $\mathcal{A}(G^v(v, \ell i))$ . It follows that  $\nu(\mathcal{A}(G(v, \ell))) = \nu(\mathcal{A}(G^v(v, \ell i)))$ . Also, elementary row operations transform

$$\mathcal{A}(G(v)) = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix} \text{ into } \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A^c & B \\ \mathbf{0} & C & D \end{pmatrix} = \mathcal{A}(G^v(v)),$$

so  $\nu(\mathcal{A}(G(v))) = \nu(\mathcal{A}(G^v(v)))$ . We conclude that  $\nu(\mathcal{A}(G^v(v, \ell i))) = \nu(\mathcal{A}(G^v(v)))$ ; according to Theorem 18 and Corollary 28, this equality implies that  $v$  is not a coloop of  $M_A(G^v(v, \ell)) = M_A(G^v)$ . ■

## 5 The principal vertex tripartition

Combining various results above, we see that if  $v \in V(G)$  then the relationships among the six adjacency matroids  $M_A(G(v))$ ,  $M_A(G(v, \ell))$ ,  $M_A(G(v, \ell i))$ ,  $M_A(G^v(v))$ ,  $M_A(G^v(v, \ell))$ ,  $M_A(G^v(v, \ell i))$  must fall into one of three cases.

**Case 1.** Suppose  $v$  is not a coloop of  $M_A(G^v(v, \ell))$ . According to part 3 of Theorem 30,  $v$  is a triple coloop of  $M_A(G(v, \ell))$ . Theorem 18 tells us that

$$Z(M_A(G^v(v))) = Z(M_A(G^v(v, \ell i))) \subset Z(M_A(G^v(v, \ell)))$$

and

$$Z(M_A(G(v))) = Z(M_A(G(v, \ell))) \subset Z(M_A(G(v, \ell i))).$$

Part 1 of Theorem 16 tells us that  $M_A(G^v(v)) = M_A(G(v))$ .  $M_A(G(v, \ell i))$  has  $v$  as a coloop, so the fact that  $v$  is not a coloop of  $M_A(G^v(v, \ell))$  implies that  $M_A(G(v, \ell i)) \neq M_A(G^v(v, \ell))$ . All in all, we have the following:



$Z(M_A(G(v, \ell i)))$  and  $Z(M_A(G^v(v, \ell)))$  are distinct nontrivial subspaces of  $2^{V(G)}$  with the same dimension, say  $\nu + 1$ ; their intersection is of dimension  $\nu$ , and

$$\begin{aligned} Z(M_A(G^v(v, \ell i))) &= Z(M_A(G^v(v))) = Z(M_A(G(v))) = Z(M_A(G(v, \ell))) \\ &= Z(M_A(G(v, \ell i))) \cap Z(M_A(G^v(v, \ell))). \end{aligned}$$

The equality  $M_A(G^v(v, \ell i)) = (M_A(G^v(v, \ell)) - v) \oplus U_{1,1}(\{v\})$  follows from Theorem 14 and Proposition 19, and  $M_A(G(v, \ell i)) = (M_A(G^v(v, \ell))/v) \oplus U_{1,1}(\{v\})$  follows from Proposition 19.

**Case 2.** Suppose  $v$  is not a coloop of  $M_A(G(v, \ell))$ . The discussion proceeds as in case 1, with  $G$  and  $G^v$  interchanged.

**Case 3.** Suppose  $v$  is a coloop of both  $M_A(G(v, \ell))$  and  $M_A(G^v(v, \ell))$ ; then parts 1 and 2 of Theorem 30 tell us that  $M_A(G(v)) = M_A(G^v(v))$ ,  $M_A(G(v, \ell)) = M_A(G^v(v, \ell))$ , and  $v$  is not a triple coloop of either  $M_A(G(v, \ell))$  or  $M_A(G^v(v, \ell))$ . Then Theorem 18 and Corollary 28 tell us that

$$\begin{aligned} Z(M_A(G(v, \ell))) &= Z(M_A(G^v(v, \ell))) = Z(M_A(G(v, \ell i))) = Z(M_A(G^v(v, \ell i))) \\ &\subset Z(M_A(G(v))) = Z(M_A(G^v(v))), \end{aligned}$$

with the dimension of the larger subspace 1 more than the dimension of the smaller. The equality  $M_A(G(v, \ell i)) = (M_A(G(v)) - v) \oplus U_{1,1}(\{v\})$  follows from Theorem 14 and Proposition 19.

To complete the proof of Theorem 20, we must verify the assertion that if  $G^v - v$  is simple, then  $v$  cannot fall under case 1 of the tripartition. Suppose  $v \in V(G)$  falls under case 1, and let

$$\mathcal{A}(G^v(v, \ell)) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A & B \\ \mathbf{0} & C & D \end{pmatrix},$$

with the first row and column corresponding to  $v$ . As  $v$  is not a coloop of  $M_A(G^v(v, \ell))$ , the first column of  $\mathcal{A}(G^v(v, \ell))$  must equal the sum of the columns corresponding to elements of some subset  $T \subseteq V(G) \setminus \{v\}$ . Consider the submatrix of  $\mathcal{A}(G^v(v, \ell))$  obtained by removing the rows and columns corresponding to vertices not in  $T \cup \{v\}$ ,

$$\mathcal{A}(G^v(v, \ell)[T \cup \{v\}]) = \begin{pmatrix} 1 & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & A' & B' \\ \mathbf{0} & C' & D' \end{pmatrix}.$$

The sum of the columns of this matrix is  $\mathbf{0}$ , so the sum of the entries of the matrix is 0; that is, the matrix has an even number of nonzero entries. As the matrix is symmetric, an even number of these nonzero entries occur off the diagonal; consequently an even number occur on the diagonal, so at least one element of  $T$  is looped in  $G^v$ .

Essentially the same argument proves that in case 2, at least one element of  $T$  is looped in  $G$ . We should point out that a garbled version of this simple

argument appeared in [24], where it was mistakenly understood to imply that there must be at least one looped vertex in  $T \cap N(v)$ . This need not be the case, as indicated by the third example in the next section. The statements of Lemma 4.4 and Corollary 4.6 of [24] should be corrected by replacing the hypothesis “if  $a$  has no looped neighbour” with “if  $H - a$  has no looped vertex.”

## 6 Three examples

Recall that if  $k < n$  then  $U_{n,k}$  denotes the  $n$ -element matroid whose circuits include all the  $(k+1)$ -element subsets of the ground set. Also,  $U_{n,n}$  denotes the free matroid on  $n$  elements, i.e.,  $\mathcal{C}(U_{n,n}) = \emptyset$ .

Let  $K_3$  be the complete graph with three vertices. Then  $M_A(K_3) \cong U_{3,2}$ .

If  $v \in V(K_3)$  then  $v$  is not a coloop of either  $M_A(K_3)$  or  $M_A(K_3^v)$ ;  $v$  falls under case 3 of the principal vertex tripartition.  $M_A(K_3(v, \ell)) = M_A(K_3(v, \ell i)) \cong U_{3,3}$ ,  $M_A(K_3^v) = M_A(K_3)$ ,  $M_A(K_3 - v) = M_A(K_3) - v \cong U_{2,2}$ , and  $M_A(K_3)/v \cong U_{2,1}$ .

Let  $K_{3\ell}$  be the graph obtained from  $K_3$  by attaching a loop to one vertex. Then  $M_A(K_{3\ell}) \cong U_{3,3}$ .

If  $v$  is one of the unlooped vertices then  $v$  is a coloop of  $M_A(K_{3\ell})$ , and  $v$  is a triple coloop of  $M_A(K_{3\ell}^v)$ ;  $v$  falls under case 2 of the principal vertex tripartition.  $M_A(K_{3\ell}(v, \ell i)) = M_A(K_{3\ell})$ ,  $M_A(K_{3\ell}(v, \ell)) \cong U_{1,1} \oplus U_{2,1}$ , and  $M_A(K_{3\ell} - v) = M_A(K_{3\ell}) - v = M_A(K_{3\ell})/v \cong U_{2,2}$ .

If  $w$  is the looped vertex then  $w$  is a coloop of  $M_A(K_{3\ell})$  and a coloop of  $M_A(K_{3\ell}^w)$ , but not a triple coloop of either;  $w$  falls under case 3 of the principal vertex tripartition.  $M_A(K_{3\ell}^w) = M_A(K_{3\ell}(w, \ell i)) = M_A(K_{3\ell})$ ,  $M_A(K_{3\ell}(w)) \cong U_{3,2}$ , and  $M_A(K_{3\ell} - w) = M_A(K_{3\ell})/w = M_A(K_{3\ell} - w) \cong U_{2,2}$ .

Let  $P_3$  be the path of length three, and  $P_{3\ell\ell}$  the graph obtained from  $P_3$  by attaching loops at the vertices of degree 1. (Equivalently,  $P_{3\ell\ell} = K_3^v$ .) Then  $M_A(P_{3\ell\ell}) \cong U_{3,2}$ .

If  $v$  is one of the looped vertices then  $v$  is not a coloop of  $M_A(P_{3\ell\ell})$ , and  $v$  is a triple coloop of  $M_A(P_{3\ell\ell}^v)$ ;  $v$  falls under case 2 of the principal vertex tripartition.  $M_A(P_{3\ell\ell}(v)) = M_A(P_{3\ell\ell}(v, \ell i)) = M_A(P_{3\ell\ell}^v) \cong U_{3,3}$ ,  $M_A(P_{3\ell\ell} - v) = M_A(P_{3\ell\ell}) - v \cong U_{2,2}$ , and  $M_A(P_{3\ell\ell})/v \cong U_{2,1}$ .

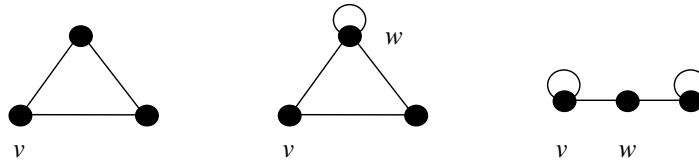


Figure 1:  $K_3$ ,  $K_{3\ell}$  and  $P_{3\ell\ell}$ .

If  $w$  is the unlooped vertex then  $w$  is not a coloop of  $M_A(P_{3\ell\ell})$  or  $M_A(P_{3\ell\ell}^w)$ ;  $w$  falls under case 3 of the principal vertex tripartition.  $M_A(P_{3\ell\ell}(w, \ell)) = M_A(P_{3\ell\ell}(w, \ell i)) \cong U_{3,3}$ ,  $M_A(P_{3\ell\ell}^w) = M_A(P_{3\ell\ell})$ ,  $M_A(P_{3\ell\ell} - w) = M_A(P_{3\ell\ell}) - w \cong U_{2,2}$  and  $M_A(P_{3\ell\ell})/w \cong U_{2,1}$ .

Observe that  $K_3$  and  $P_{3\ell\ell}$  have isomorphic adjacency matroids, and their principal vertex tripartitions are distinct. On the other hand,  $K_{3\ell}$  and  $P_{3\ell\ell}$  have nonisomorphic adjacency matroids and equivalent principal vertex tripartitions.

## 7 Set systems and $\Delta$ -matroids

In this section we briefly summarize a number of definitions and results related to set systems and  $\Delta$ -matroids. The reader is referred to [8] – [13] for detailed discussions.

### 7.1 Set systems

A *set system* (over  $V$ ) is a tuple  $D = (V, \sigma)$  with  $V$  a finite set, called the *ground set*, and  $\sigma$  a family of subsets of  $V$ . We often write  $Y \in D$  to mean  $Y \in \sigma$ . A set system  $D$  is called *proper* if  $\sigma \neq \emptyset$ , and *normal* if  $\emptyset \in D$ . Let  $X \subseteq V$ . If  $D$  is proper, then we define the *distance* between  $X \subseteq V$  and  $D$  by  $d_D(X) = \min(\{|X \Delta Y| \mid Y \in D\})$ . Moreover, we let  $d_D = d_D(\emptyset)$ , so that  $D$  is normal if and only if  $d_D = 0$ . We define the *restriction* of  $D$  to  $X$  by  $D[X] = (X, \sigma')$  where  $\sigma' = \{Y \in \sigma \mid Y \subseteq X\}$ , and the *deletion* of  $X$  from  $D$  by  $D - X = D[V - X]$ . Let  $\min(\sigma)$  ( $\max(\sigma)$ , resp.) denote the family of minimal (maximal, resp.) sets in  $\sigma$  with respect to set inclusion, and let  $\min(D) = (V, \min(\sigma))$  ( $\max(D) = (V, \max(\sigma))$ , resp.) be the corresponding set systems. A set system  $D$  is *equicardinal* if for all  $X_1, X_2 \in D$ ,  $|X_1| = |X_2|$ .

Let  $D$  again be a set system. For  $X \subseteq V$  we define the *pivot* (also called *twist* in the literature [8]) by  $D * X = (V, \sigma * X)$ , where  $\sigma * X = \{Y \Delta X \mid Y \in \sigma\}$ . Also, if  $v \in V$  we define the *contraction* of  $D/v$  by  $D/v = D * v - v$  [10, Property 2.1]. Note that  $d_D(X) = d_{D*X}$ ; in particular,  $D * X$  is normal if and only if  $X \in D$ . Also, note that  $D * V$  is obtained from  $D$  by complementing every set of  $D$  with respect to the ground set. Thus, it is easy to see that  $\min(D) = \max(D * V) * V$  and  $\max(D) = \min(D * V) * V$ . For  $X \subseteq V$  we define *loop complementation* by  $D + X = (V, \sigma')$ , where  $Y \in \sigma'$  iff  $|\{Z \in D \mid (Y \setminus X) \subseteq Z \subseteq Y\}|$  is odd [11]. In particular, if  $v \in V$  then  $D + v = (V, \sigma')$  with  $\sigma' = \{Y \mid v \notin Y \in D\} \cup \{Y \cup \{v\} \mid Y \in D \text{ and } Y \cup \{v\} \notin D\}$ . For  $X \subseteq V$  we define the *dual pivot* by  $D * X = D + X * X + X$ . It turns out that  $D * X = (V, \sigma')$ , where  $Y \in \sigma'$  iff  $|\{Z \in D \mid Y \subseteq Z \subseteq Y \cup X\}|$  is odd. It is easy to verify that  $\max(D) = \max(D * X)$  for all  $X \subseteq V$ .

For convenience, we often write  $D - \{v\}$ ,  $D * \{v\}$ ,  $D \bar{*} \{v\}$  etc. simply as  $D - v$ ,  $D * v$ ,  $D \bar{*} v$  etc. Also, we assume left-associativity of the operations. E.g.,  $D \bar{*} v * w - v$  denotes  $((D \bar{*} v) * w) - v$ . Deletion, pivot, loop complementation, and dual pivot belong to a class of operations called vertex flips which commute on

different elements (see [11]). For example, for  $v, w \in V$  and  $v \neq w$ ,  $D - v * w = D * w - v$ ,  $D \bar{*} v * w = D * w \bar{*} v$  and  $D \bar{*} v \bar{*} w = D \bar{*} w \bar{*} v$ . Moreover, pivot, loop complementation, and dual pivot are involutions.

Suppose  $D = (V, \sigma)$  is a set system, and  $v \in V$ . Then  $\sigma = \sigma' \cup \sigma''$ , where  $Y \in \sigma'$  (resp.  $Y \in \sigma''$ ) iff  $v \notin Y \in \sigma$  (resp.  $v \in Y \in \sigma$ ). We define  $D \bar{\sim} v = (V, \sigma')$  and  $D \tilde{/} v = (V, \sigma'')$ . That is,  $D \bar{\sim} v$  is the set system on  $V$  that includes the same sets as  $D - v$ , and  $D \tilde{/} v$  is the set system on  $V$  that includes the sets  $Y \cup \{v\}$  with  $Y \in D/v$ .

**Theorem 32** *Let  $D$  be a set system on  $V$ , and suppose  $v \in V$  has the property that  $D \bar{\sim} v$  is a proper set system. We have*

$$\max(D \bar{\sim} v + v) = (\max(D * v)) \tilde{/} v.$$

*That is,  $\max(D \bar{\sim} v + v)$  is obtained from  $\max(D * v)$  by removing the sets that do not contain  $v$ .*

**Proof.** By definition,  $D \bar{\sim} v + v = (V, \sigma)$  where  $\sigma = \{Y, Y \cup \{v\} \mid v \notin Y \in D\}$ . Consequently all sets in  $\max(D \bar{\sim} v + v)$  contain  $v$ ; that is,  $\max(D \bar{\sim} v + v) = \max((D \bar{\sim} v + v) \tilde{/} v)$ . Moreover, the family of sets of  $D \bar{\sim} v + v$  that include  $v$  is  $\{Y \cup \{v\} \mid v \notin Y \in D\}$ , which is equal to the family of sets of  $D * v$  that include  $v$ . That is,  $(D \bar{\sim} v + v) \tilde{/} v = (D * v) \tilde{/} v$ ; it follows immediately that  $\max((D \bar{\sim} v + v) \tilde{/} v) = \max((D * v) \tilde{/} v)$ . The equality  $(\max((D * v)) \tilde{/} v = \max((D * v) \tilde{/} v)$  is obvious; the maximal elements of  $D * v$  that contain  $v$  are the elements of  $D * v$  that are maximal among those that contain  $v$ . ■

Theorem 32 is the first of several results that extend properties of adjacency matroids to general set systems or  $\Delta$ -matroids. As we discuss in Theorem 45 below, this theorem extends part of Proposition 19 of the introduction.

## 7.2 $\Delta$ -matroids

A *delta-matroid* ( $\Delta$ -matroid for short) is a proper set system  $D$  that satisfies the *symmetric exchange axiom*: For all  $X, Y \in D$  and all  $u \in X \Delta Y$ ,  $X \Delta \{u\} \in D$  or there is a  $v \in X \Delta Y$  with  $v \neq u$  such that  $X \Delta \{u, v\} \in D$  (or both) [8]. A proper set system  $D$  is a  $\Delta$ -matroid if and only if for each  $X \subseteq V$ ,  $\min(D * X)$  is equicardinal (see [12]). Equivalently,  $D$  is a  $\Delta$ -matroid if and only if for each  $X \subseteq V$ ,  $\max(D * X)$  is equicardinal. Let  $D$  be a  $\Delta$ -matroid and  $v \in V$ . Then  $D * v$  is a  $\Delta$ -matroid. Moreover,  $D - v$  is a  $\Delta$ -matroid if and only if  $D - v$  is proper.

However  $D \bar{*} v$  may be a proper set system without being a  $\Delta$ -matroid. As in [11, Example 10], let  $V$  be a finite set with  $|V| \geq 3$ , and consider the  $\Delta$ -matroid  $D = (V, \sigma)$  with  $\sigma = 2^V \setminus \{\emptyset\}$ . Then it is easy to see that the symmetric exchange axiom does not hold for  $D \bar{*} V = (V, \{\emptyset, V\})$ .

If we assume a matroid  $M$  is described by its family of bases, i.e.,  $M$  is the set system  $(V, B)$  where  $B$  is the set of bases of  $M$ , then it is shown in [9, Proposition 3] that a matroid  $M$  is precisely an equicardinal  $\Delta$ -matroid. Moreover, a

proper set system  $D$  is a  $\Delta$ -matroid if and only if for each  $X \subseteq V$ ,  $\max(D * X)$  is a matroid [10, Property 4.1]. Note that for a matroid  $M$  (described by its family of bases),  $M * V$  is the dual matroid of  $M$ . Hence,  $D$  is a  $\Delta$ -matroid if and only if for each  $X \subseteq V$ ,  $\min(D * X)$  is a matroid. Clearly for any  $\Delta$ -matroid  $D$ ,  $r(\min(D)) = d_D$  and  $\nu(\max(D)) = d_D(V)$ , where  $r$  and  $\nu$  denote the rank and nullity of a matroid respectively. The deletion operation of  $\Delta$ -matroids coincides with the deletion operation of matroids only for non-coloops. Also, the contraction operation of  $\Delta$ -matroids coincides with the contraction operation of matroids only for non-loops. Fortunately, as deletion and contraction for matroids coincide for both loops and coloops, matroid-deletion of a coloop is equal to  $\Delta$ -matroid-contraction of that element, and matroid-contraction of a loop is equal to  $\Delta$ -matroid-deletion of that element.

We will need Theorem 14 from [12] (the original formulation is in terms of rank rather than nullity).

**Proposition 33** *Let  $D$  be a  $\Delta$ -matroid, and suppose  $v \in V$  has the property that  $D + v$  is also a  $\Delta$ -matroid. Then  $\max(D)$ ,  $\max(D * v)$ , and  $\max(D + v)$  are matroids such that precisely two of the three are equal, to say  $D_1$ . Moreover the third,  $D_2$ , has  $(D_2 - v) \oplus U_{1,1}(\{v\}) = D_1$  and  $\nu(D_2) = \nu(D_1) + 1$ .*

Note that consequently,  $\nu(\max(D)) = \nu(\max(D * v))$  if and only if  $\max(D) = \max(D * v)$ . Also, Theorem 32 tells us that if  $D \sim v$  is proper, then  $\max(D \sim v + v)$  can replace  $\max(D * v)$  in Proposition 33: if  $\max(D * v) = D_1$  then  $v$  is a coloop of  $\max(D * v) = (\max(D * v)) \sim v = \max(D \sim v + v)$ , and if  $\max(D * v) = D_2$  then  $\max(D * v)$  and  $(\max(D * v)) \sim v = (\max(D * v)/v) \oplus U_{1,1}(\{v\})$  are different matroids ( $v$  is a coloop in the latter but not the former) with the same nullity.

### 7.3 Representing graphs by $\Delta$ -matroids

Let  $G = (V, E)$  be a graph. We define  $\mathcal{D}_G$  to be the set system  $(V, \sigma)$  where  $\sigma = \{X \subseteq V \mid \mathcal{A}(G)[X] \text{ is nonsingular over } GF(2)\}$ . It is shown in [8] that  $\mathcal{D}_G$  is a normal  $\Delta$ -matroid (by convention, the empty matrix is nonsingular). Moreover, if  $G$  is a looped simple graph then given  $\mathcal{D}_G$ , one can (re)construct  $G$ :  $\{u\}$  is a loop in  $G$  if and only if  $\{u\} \in \mathcal{D}_G$ , and  $\{u, v\}$  is an edge in  $G$  if and only if  $(\{u, v\} \in \mathcal{D}_G) \Delta ((\{u\} \in \mathcal{D}_G) \wedge (\{v\} \in \mathcal{D}_G))$ , see [10, Property 3.1]. In this way, the family of looped simple graphs with vertex-set  $V$  can be considered as a subset of the family of  $\Delta$ -matroids on the ground set  $V$ .

It is shown in [12, Theorem 16] that, for all  $X \subseteq V$ ,  $d_{\mathcal{D}_G}(X) = \nu(\mathcal{A}(G)[X])$ , where  $\nu$  denotes  $GF(2)$ -nullity. If  $v$  is a looped vertex of  $G$ , then it is shown in [18] that  $\mathcal{D}_G * v$  represents the graph  $G^v$ . Moreover, if  $v$  is a unlooped vertex of  $G$ , then  $\mathcal{D}_G \bar{*} v$  represents the graph  $G^v$  (see [11]). In this way,  $G^v$  may be defined using  $\Delta$ -matroids. However,  $\mathcal{D}_G * v$  on an unlooped vertex  $v$  and  $\mathcal{D}_G \bar{*} v$  on a looped vertex  $v$  do *not* represent graphs in general:

**Proposition 34 (Remark below Theorem 22 in [12])** *Let  $G$  be a graph, and  $\varphi$  be any sequence of pivot, dual pivot and loop complement operations on*

elements of  $V(G)$ . Then  $\emptyset \in (\mathcal{D}_G)\varphi$  if and only if  $(\mathcal{D}_G)\varphi = \mathcal{D}_{G'}$  for some graph  $G'$ .

In contrast,  $\max((\mathcal{D}_G)\varphi)$  does always have a graph representation.

**Theorem 35** *Let  $G$  be a graph, and let  $\varphi$  be any sequence of pivot, dual pivot and local complement operations on elements of  $V(G)$ . Then  $\max((\mathcal{D}_G)\varphi)$  is a binary matroid.*

**Proof.** Let  $D = (\mathcal{D}_G)\varphi$ . Let  $X \in \min(D)$ . Then  $\emptyset \in D\bar{*}X$  if and only if  $|\{Z \in D \mid Z \subseteq X\}|$  is odd. Since  $\{Z \in D \mid Z \subseteq X\} = \{X\}$  by definition of  $X$ , we have  $\emptyset \in D\bar{*}X = (\mathcal{D}_G)\varphi\bar{*}X$ . By Proposition 34,  $(\mathcal{D}_G)\varphi\bar{*}X = \mathcal{D}_{G'}$  for some graph  $G'$ . Thus,  $\max((\mathcal{D}_G)\varphi) = \max((\mathcal{D}_G)\varphi\bar{*}X) = \max(\mathcal{D}_{G'})$  and we are done. ■

For convenience, we define the *pivot* of a vertex  $v$  on a graph  $G$ , denoted  $G * v$ , by  $G^v$  if  $v$  is looped, and it is not defined otherwise. Similarly, we define the *dual pivot* of vertex  $v$  on  $G$ , denoted  $G\bar{*}v$ , by  $G^v$  if  $v$  is unlooped, and it is not defined otherwise. For a graph, *loop complementation* of a vertex  $v \in V(G)$ , denoted by  $G + v$ , toggles the existence of a loop on  $v$ . I.e.,  $v$  is a looped vertex of  $G$  iff  $v$  is not a looped vertex of  $G + v$ . It is shown in [11] that  $\mathcal{D}_{G+v} = \mathcal{D}_G + v$  (i.e., loop complementation for  $\Delta$ -matroids generalizes loop complementation for graphs).

It is easy to verify that for each  $v \in V(G)$ ,  $\mathcal{D}_{G-v} = \mathcal{D}_G - v$ . Theorem 24 of the introduction follows readily from this easy observation and the strong principal minor theorem [21] (see also Theorem 26 above). As we will see in the following sections, the equality

$$\max(\mathcal{D}_G) = M_A(G)$$

(where  $M_A(G)$  is described by its family of bases) allows us to give various results stated in the introduction completely different proofs, using  $\Delta$ -matroids rather than linear algebra over  $GF(2)$ .

## 8 Deletion/contraction and min/max for $\Delta$ -matroids

In this section we show, under the assumption of some mild conditions, that both contraction and deletion commute with both the min and the max operation for  $\Delta$ -matroids. In fact, some results hold for set systems in general. We will apply these results to graphs in the next section.

Let  $D$  be a set system. The notions of loop and coloop for matroids (described by their families of bases) may be directly generalized to set systems. An element  $v \in V$  is called a *coloop* of  $D$  if  $v \in X$  for each  $X \in D$ . Clearly,  $v$  is a coloop of  $D$  if and only if  $D - v$  is not proper. Similarly,  $v \in V$  is called a *loop* of  $D$  if  $v$  is a coloop of  $D * v$ , i.e.,  $v \notin X$  for each  $X \in D$ .

We first show that the min operation and the deletion operation on an element  $v$  commute for proper set systems  $D$ , provided that  $D - v$  is proper. Note

that  $D - v$  is proper, i.e.,  $v$  is not a coloop of  $D$ , if and only if  $v$  is not a coloop of  $\min(D)$ .

**Theorem 36** *Let  $D$  be a proper set system, and let  $v \in V$  such that  $D - v$  is proper. Then  $\min(D) - v = \min(D - v)$ .*

**Proof.** Since  $D - v$  is proper,  $\min(D - v)$  is well defined. Let  $X \in \min(D) - v$ . Then  $X \in \min(D)$  and  $v \notin X$ . Hence,  $X \in \min(D - v)$ . Conversely, if  $X \in \min(D - v)$ , then  $X \in D$  and  $v \notin X$ . Let  $Y \subseteq X$  with  $Y \in \min(D)$ . Then clearly,  $v \notin Y$  and thus  $Y \in \min(D - v)$ . Hence  $X = Y$  and  $X \in \min(D)$ . Therefore,  $X \in \min(D) - v$ . ■

Next, we show that the max operation and the deletion operation on an element  $v$  commute for  $\Delta$ -matroids  $D$ , provided that  $v$  is not a coloop of  $\max(D)$ .

**Theorem 37** *Let  $D$  be a  $\Delta$ -matroid, and let  $v \in V$  such that  $v$  is not a coloop of  $\max(D)$ . Then  $\max(D) - v = \max(D - v)$ .*

**Proof.** Since  $v$  is not a coloop of  $\max(D)$ , there is a  $Z \in \max(D)$  with  $v \notin Z$ . Therefore  $D - v$  is proper, and so  $\max(D - v)$  is well defined. Let  $X \in \max(D) - v$ . Then  $X \in \max(D)$  and  $v \notin X$ . Hence,  $X \in \max(D - v)$ . Conversely, if  $X \in \max(D - v)$ , then  $X \in D$  and  $v \notin X$ , and  $X$  is maximal with this property. As  $v$  is not a coloop of  $\max(D)$ , there is a  $Z \in \max(D)$  with  $v \notin Z$ . Therefore,  $Z \in \max(D) - v$ . By the first part of this proof,  $Z \in \max(D - v)$ . Now, as  $D - v$  is a  $\Delta$ -matroid,  $\max(D - v)$  is equicardinal and so  $|Z| = |X|$ . Moreover, since  $D$  is a  $\Delta$ -matroid,  $\max(D)$  is equicardinal, therefore  $X \in \max(D)$  and so,  $X \in \max(D) - v$ . ■

The next example illustrates that Theorem 37 does not hold for set systems in general. This in contrast with Theorem 36, which *does* hold for set systems in general.

**Example 38** *Let  $D = (V, D)$  be a set system with  $V = \{u, v, w\}$  and  $D = \{\{u\}, \{v\}, \{v, w\}\}$ . Then  $w$  is not a coloop of  $\max(D) = (V, \{\{u\}, \{v, w\}\})$ , and  $\max(D) - w = (\{u, v\}, \{\{u\}\})$  while  $\max(D - w) = (\{u, v\}, \{\{u\}, \{v\}\})$ .*

We formulate now the max (min, resp.) “counterparts” of Theorem 36 (Theorem 37, resp.). These results show that contraction commutes with the min and max operations.

**Theorem 39** *Let  $D$  be a proper set system and  $v \in V$ .*

1. *If  $v$  is not a loop of  $D$ , then  $\max(D) * v - v = \max(D * v - v)$ .*
2. *If  $D$  is moreover a  $\Delta$ -matroid and  $v$  is not a loop of  $\min(D)$ , then  $\min(D) * v - v = \min(D * v - v)$ .*

**Proof.** We start by showing the first result. We have  $\max(D) * v - v = \min(D * V) * V * v - v = \min(D * V) - v * (V \setminus \{v\})$ . Now,  $v$  is not a coloop of  $D * V$ . Thus,  $D * V - v$  is proper. By Theorem 36,  $\min(D * V) - v * (V \setminus \{v\}) = \min(D * (D * V - v))$ .

$V-v)*(V\setminus\{v\}) = \min(D*v-v*(V\setminus\{v\}))* (V\setminus\{v\}) = \max(D*v-v)$ . The proof of the second result is essentially identical to that of the first result. We have  $\min(D)*v-v = \max(D*V)*V*v-v = \max(D*V)-v*(V\setminus\{v\})$ . Now,  $v$  is not a coloop of  $\min(D)*V = \max(D*V)$ . By Theorem 37,  $\max(D*V)-v*(V\setminus\{v\}) = \max(D*V-v)*(V\setminus\{v\}) = \max(D*v-v*(V\setminus\{v\}))* (V\setminus\{v\}) = \min(D*v-v)$ . ■

## 9 From $\Delta$ -matroids to graphs

In this section we use results of Sections 7 and 8 to give new proofs of several theorems about adjacency matroids stated earlier in the paper. These proofs are fundamentally different from the earlier ones, as they are combinatorial and do not involve matrices. Recall that if  $G$  is a graph then  $\mathcal{D}_G$  is a  $\Delta$ -matroid with  $M_A(G) = \max(\mathcal{D}_G)$ ;  $\mathcal{D}_G$  is normal, so no  $v \in V$  is a coloop of  $\mathcal{D}_G$ .

The following three results are quite straightforward consequences of the fact that for  $\mathcal{D}_G$ , max commutes with both deletion (of non-coloops) and contraction (of non-loops), cf. Theorems 37 and 39.

**Theorem 40** *If  $v$  is not a coloop of  $M_A(G)$ , then  $M_A(G) - v = M_A(G - v)$ .*

**Proof.** If  $v$  is not a coloop of  $M_A(G)$ , then  $M_A(G) - v = \max(\mathcal{D}_G) - v$ . By Theorem 37,  $\max(\mathcal{D}_G) - v = \max(\mathcal{D}_G - v) = \max(\mathcal{D}_{G-v}) = M_A(G - v)$ . ■

**Theorem 41** *If  $v \in V(G)$  is a looped vertex, then  $M_A(G)/v = M_A(G^v - v)$ .*

**Proof.** Since  $\{v\} \in \mathcal{D}_G$ ,  $v$  is not a loop of  $\mathcal{D}_G$ . Consequently,  $v$  is not a loop of  $\max(\mathcal{D}_G)$ . We have therefore  $M_A(G)/v = \max(\mathcal{D}_G) * v - v$ . By Theorem 39,  $\max(\mathcal{D}_G) * v - v = \max(\mathcal{D}_G * v - v)$ . Since  $v$  is a looped vertex,  $\mathcal{D}_G * v = \mathcal{D}_{G*v}$ , and thus  $\max(\mathcal{D}_G * v - v) = \max(\mathcal{D}_{G*v-v}) = M_A(G * v - v)$ . The result follows as  $G * v = G^v$ . ■

**Theorem 42** *Suppose  $v$  is an unlooped vertex of  $G$ .*

1. *If  $v$  is isolated, then  $M_A(G)/v = M_A(G - v)$ .*
2. *If  $w$  is an unlooped neighbour of  $v$ , then  $M_A(G)/v = M_A((G^w)^v - v)$ .*
3. *If  $w$  is a looped neighbour of  $v$ , then  $M_A(G)/v = M_A(((G^v)^w)^v - v)$ .*

**Proof.** We first prove Result 1. If  $v$  is isolated and unlooped, then  $v$  is a loop of  $M_A(G)$ . Hence,  $M_A(G)/v = M_A(G) - v$ . Moreover,  $v$  is not a coloop of  $M_A(G)$ . The result follows now by Theorem 40.

We now prove Results 2 and 3. Let  $w$  be a neighbour of  $v$ . As  $\{v, w\} \in \mathcal{D}_G$ ,  $v$  is not a loop of  $\mathcal{D}_G$ . Hence,  $M_A(G)/v = \max(\mathcal{D}_G) * v - v$ . By Theorem 39,  $\max(\mathcal{D}_G) * v - v = \max(\mathcal{D}_G * v - v)$ . Now,  $\max(\mathcal{D}_G * v - v) = \max(\mathcal{D}_G * v - v \bar{*} w) = \max(\mathcal{D}_G \bar{*} w * v - v)$ . On the one hand, if  $w$  is unlooped, then it is easy to verify that  $G \bar{*} w * v$  is defined. Hence  $\mathcal{D}_G \bar{*} w * v = \mathcal{D}_{G \bar{*} w * v}$ . Finally,



$\max(\mathcal{D}_{G^*w^*v} - v) = M_A(G^*w^*v - v) = M_A((G^w)^v - v)$ . This proves Result 2. On the other hand, if  $w$  is looped, then it is easy to verify that  $G^*v^*w^*v$  is defined. Hence  $\mathcal{D}_{G^*v^*w^*v} = \mathcal{D}_{G^*v^*w^*v}$ . Finally,  $\max(\mathcal{D}_{G^*w^*v} - v) = \max(\mathcal{D}_{G^*v^*w^*v} - v) = M_A(G^*v^*w^*v - v) = M_A(((G^v)^w)^v - v)$ . This proves Result 3. ■

The next result is obtained from Proposition 33.

- Theorem 43** 1. If  $v \in V(G)$  is unlooped, then  $M_A(G^v) = M_A(G)$ .
2. If  $v \in V(G)$  is a coloop of both  $M_A(G)$  and  $M_A(G^v)$ , then  $M_A(G^v) = M_A(G)$ .
3. If  $v \in V(G)$  is looped and not a coloop of one of  $M_A(G)$ ,  $M_A(G^v)$ , then  $v$  is a coloop of the other and  $M_A(G^v)$  and  $M_A(G)$  are of different ranks.

**Proof.** We first show Result 1. If  $v$  is unlooped, then  $G^v = G^*v$ . Thus,  $M_A(G) = \max(\mathcal{D}_G) = \max(\mathcal{D}_{G^*v}) = \max(\mathcal{D}_{G^*v}) = M_A(G^v)$  and the result follows.

We now show Results 2 and 3. If  $v$  is unlooped, then we are done by Result 1. So, assume  $v$  is looped. Then we have  $G^v = G^*v$ . The result follows now by Proposition 33. ■

**Theorem 44** If  $v \in V(G)$ , then  $M_A(G) - v = M_A(G^v) - v$ .

**Proof.** If  $v$  is an unlooped vertex, then by Theorem 43.1,  $M_A(G) = M_A(G^v)$  and the equality holds. Assume now that  $v$  is a looped vertex. By Theorem 43.3,  $v$  is a coloop of at least one of  $M_A(G)$  and  $M_A(G^v)$ . If  $v$  is a coloop of both  $M_A(G)$  and  $M_A(G^v)$ , then the equality holds by Theorem 43.2.

We assume now without loss of generality that  $v$  is a coloop of  $M_A(G)$  and  $v$  is not a coloop of  $M_A(G^v)$  (the other case follows by considering graph  $G := G^v$ ). We have in this case  $M_A(G) - v = M_A(G)/v$ . By Theorem 41,  $M_A(G)/v = M_A(G^v - v)$ . As  $v$  is not a coloop of  $M_A(G^v)$ , by Theorem 40,  $M_A(G^v - v) = M_A(G^v) - v$  and the result follows. ■

By the way, the interested reader will have no trouble using [12, Theorem 15] to prove Theorems 30 and 31.

It is easy to see that  $\mathcal{D}_G \widetilde{-} v + v = \mathcal{D}_{G(v, li)}$ . Hence we obtain the following corollary to Theorem 32 and Proposition 33. Part 1 follows from part 2, which is part of Proposition 19; and part 3 includes some of the assertions of Theorem 20.

**Theorem 45** If  $G$  is a graph with a looped vertex  $v$ , then the following hold.

1.  $\nu(M_A(G^v)) = \nu(M_A(G(v, li)))$ .
2.  $\mathcal{B}(M_A(G(v, li))) = \{B \in \mathcal{B}(M_A(G^v)) \mid v \in B\}$ , or equivalently  $M_A(G(v, li)) = (M_A(G^v)/v) \oplus U_{1,1}(\{v\})$ .
3.  $M_A(G(v, li)) = M_A(G^v)$  iff  $\nu(M_A(G)) \leq \nu(M_A(G^v))$ .

## 10 The interlace and Tutte polynomials

In this section we discuss the connection between the interlace polynomials of a graph  $G$ , introduced by Arratia, Bollobás and Sorkin [2, 3, 4], and the Tutte polynomials of the adjacency matroids of  $G$  and its subgraphs. (The Tutte polynomial is described by many authors; see [6] and [27] for instance. Especially thorough accounts are given by Brylawski and Oxley [31] and Ellis-Monaghan and Merino [15].) In particular, we show that the fundamental recursion of the two-variable interlace polynomial may be derived from properties of the Tutte polynomial.

**Definition 46** *Let  $G$  be a graph. Then the interlace polynomial of  $G$  is*

$$\begin{aligned} q(G) &= \sum_{S \subseteq V(G)} (x-1)^{|S|-\nu(\mathcal{A}(G[S]))} (y-1)^{\nu(\mathcal{A}(G[S]))} \\ &= \sum_{S \subseteq V(G)} (x-1)^{|S|} \cdot \left( \frac{y-1}{x-1} \right)^{\nu(\mathcal{A}(G[S]))}, \end{aligned}$$

where  $\nu$  denotes  $GF(2)$ -nullity.

Arratia, Bollobás and Sorkin [4] showed that  $q(G)$  may also be defined recursively:

1. If  $v$  is a looped vertex of  $G$  then  $q(G) = q(G-v) + (x-1)q(G^v-v)$ .
2. If  $v$  and  $w$  are unlooped neighbours in  $G$  then  $q(G) = q(G-v) + q(((G^v)^w)^v-v) + ((x-1)^2-1) \cdot q(((G^v)^w)^v-v-w)$ .
3. If  $G$  consists solely of unlooped vertices then  $q(G) = y^{|V(G)|}$ .

So far, our discussion of matroids has been focused on their circuits. Here are two other basic definitions of matroid theory.

**Definition 47** *Let  $M$  be a matroid on a set  $V$ . A subset  $I \subseteq V$  is independent if  $I$  contains no circuit of  $M$ . The rank of a subset  $S \subseteq V$  is the cardinality of the largest independent set(s) in  $S$ ; it is denoted  $r(S)$ .*

All the notions of matroid theory can be equivalently defined from the independent sets or the rank function, instead of the circuits. For instance, Definitions 11 and 10 are equivalent to: if  $M$  is a matroid on a set  $V$  and  $v \in V$  then the contraction  $M/v$  and the deletion  $M-v$  are the matroids on  $V \setminus \{v\}$  with the rank functions  $r_{M/v}(S) = r(S \cup \{v\}) - r(\{v\})$  and  $r_{M-v}(S) = r(S)$ .

Recall that if  $G$  is a graph, then the circuits of  $M_A(G)$  are the minimal nonempty subsets  $S \subseteq V(G)$  such that the columns of  $\mathcal{A}(G)$  corresponding to elements of  $S$  are linearly dependent. It follows that the rank in  $M_A(G)$  of a subset  $S \subseteq V(G)$  is simply the  $GF(2)$ -rank of the  $|V(G)| \times |S|$  submatrix of  $\mathcal{A}(G)$  obtained by removing the columns corresponding to vertices not in  $S$ . This submatrix of  $\mathcal{A}(G)$  is obtained from  $\mathcal{A}(G[S])$  by adjoining rows corresponding to vertices not in  $S$ , so  $r(S) \geq |S| - \nu(\mathcal{A}(G[S]))$ . The difference between  $r(S)$  and  $|S| - \nu(\mathcal{A}(G[S]))$  varies with  $G$  and  $S$ , in general; however if  $r(S) = r(M_A(G))$  then according to the strong principal minor theorem (see [21] or Theorem 26),  $r(S) = |S| - \nu(\mathcal{A}(G[S]))$ .

**Definition 48** Let  $M$  be a matroid on a set  $V$ . Then the Tutte polynomial of  $M$  is

$$t(M) = \sum_{S \subseteq V} (x-1)^{r(V)-r(S)} (y-1)^{|S|-r(S)}.$$

The equations  $r_{M/v}(S) = r(S \cup \{v\}) - r(\{v\})$  and  $r_{M-v}(S) = r(S)$  imply that the Tutte polynomial may be calculated recursively using the following steps.

1. If  $v$  is a loop of  $M$  then  $t(M) = y \cdot t(M-v)$ .
2. If  $v$  is a coloop of  $M$  then  $t(M) = x \cdot t(M/v)$ .
3. If  $v$  is neither a loop nor a coloop, then  $t(M) = t(M/v) + t(M-v)$ .
4.  $t(\emptyset) = 1$ .

The distinction between  $M-v$  and  $M/v$  in steps 1 and 2 is traditional, but for us it is unimportant as Definitions 10 and 11 have  $M/v = M-v$  for loops and coloops.

We single out the leading term of  $t(M)$  (the term corresponding to  $S = V$ ) for special attention.

**Definition 49** Let  $M$  be a matroid on a set  $V$ . Then  $\lambda_M(y) = (y-1)^{|V|-r(V)}$ .

Like  $t(M)$ ,  $\lambda_M$  has a recursive description derived from the equations  $r_{M/v}(S) = r(S \cup \{v\}) - r(\{v\})$  and  $r_{M-v}(S) = r(S)$ :

**Proposition 50** 1. If  $v$  is a loop of  $M$  then  $\lambda_M = (y-1) \cdot \lambda_{M-v} = (y-1) \cdot \lambda_{M/v}$ .  
 2. If  $v$  is a coloop of  $M$  then  $\lambda_M = \lambda_{M-v} = \lambda_{M/v}$ .  
 3. If  $v$  is neither a loop nor a coloop, then  $\lambda_M = (y-1) \cdot \lambda_{M-v} = \lambda_{M/v}$ .  
 4.  $\lambda_\emptyset = 1$ .

If  $G$  is a graph then we adopt the abbreviated notation  $\lambda_{M_A(G)} = \lambda_G$ .

**Corollary 51** 1. If  $v$  is an unlooped vertex of  $G$  then  $\lambda_G = \lambda_{G^v}$ .

2. If  $v$  is a looped vertex of  $G$  then  $\lambda_G = \lambda_{G^v-v}$ .
3. If  $v$  is an isolated, unlooped vertex of  $G$  then  $\lambda_G = (y-1) \cdot \lambda_{G-v}$ .
4.  $\lambda_\emptyset = 1$ .

**Proof.** If  $v$  is an unlooped vertex of  $G$  then  $M_A(G) = M_A(G^v)$  by Theorem 16. If  $v$  is a looped vertex of  $G$  then  $v$  is not a loop of  $M_A(G)$ , so  $\lambda_{M_A(G)} = \lambda_{M_A(G)/v} = \lambda_{G^v-v}$  by Proposition 50 and Theorem 12. If  $v$  is an isolated, unlooped vertex of  $G$  then  $v$  is a loop of  $M_A(G)$ , so  $\lambda_{M_A(G)} = (y-1) \cdot \lambda_{M_A(G)-v} = (y-1) \cdot \lambda_{G-v}$  by Theorem 14 and Proposition 50. ■

If  $G$  is a graph then Definitions 46 and 49 imply that

$$q(G) = \sum_{S \subseteq V(G)} (x-1)^{|S|} \cdot \lambda_{G[S]} \left(1 + \frac{y-1}{x-1}\right) \quad (1)$$

and hence for each  $v \in V(G)$ ,

$$q(G) - q(G-v) = \sum_{v \in S \subseteq V(G)} (x-1)^{|S|} \cdot \lambda_{G[S]} \left(1 + \frac{y-1}{x-1}\right). \quad (2)$$

Suppose  $v$  is a looped vertex and  $v \in S \subseteq V(G)$ . Then Corollary 51 tells us that  $\lambda_{G[S]} = \lambda_{G[S]^{v-v}}$ . Clearly  $G[S]^v - v = G^v[S] - v = (G^v - v)[S \setminus \{v\}]$ , so it follows from equations (1) and (2) that

$$\begin{aligned} q(G) - q(G - v) &= (x - 1) \cdot \sum_{v \in S \subseteq V(G)} (x - 1)^{|S|-1} \cdot \lambda_{G^v[S]-v} \left(1 + \frac{y-1}{x-1}\right) \\ &= (x - 1) \cdot \sum_{S \subseteq V(G) \setminus \{v\}} (x - 1)^{|S|} \cdot \lambda_{(G^v-v)[S]} \left(1 + \frac{y-1}{x-1}\right) \\ &= (x - 1) \cdot q(G^v - v). \end{aligned}$$

This yields the first formula of the recursive description of  $q$ .

Suppose now that  $v$  is an unlooped vertex of  $G$ , and  $w$  is an unlooped neighbour of  $v$  in  $G$ . Let  $H = (G^v)^w$ ; then  $v$  and  $w$  are looped neighbours in  $H$ . Equation (2) tells us that

$$q(H^v - v) - q(H^v - v - w) = \sum_{w \in S \subseteq V(H^v - v)} (x - 1)^{|S|} \cdot \lambda_{H^v[S]} \left(1 + \frac{y-1}{x-1}\right).$$

Suppose  $w \in S \subseteq V(H^v - v)$ ; obviously then  $H^v[S] = H^v[S \cup \{v\}] - v = H[S \cup \{v\}]^v - v$ . As  $v$  and  $w$  are both looped in  $H[S \cup \{v\}] = (G^v)^w[S \cup \{v\}]$ , Corollary 51 tells us that

$$\lambda_{H^v[S]} = \lambda_{H[S \cup \{v\}]^v - v} = \lambda_{H[S \cup \{v\}]} = \lambda_{(G^v)^w[S \cup \{v\}]} = \lambda_{((G^v)^w[S \cup \{v\}])^w - w}.$$

Note that  $((G^v)^w[S \cup \{v\}])^w - w = ((G^v)^w)^w[S \cup \{v\}] - w = G^v[(S \setminus \{w\}) \cup \{v\}] = G[(S \setminus \{w\}) \cup \{v\}]^v$ . As  $v$  is unlooped in  $G[(S \setminus \{w\}) \cup \{v\}]$ , Corollary 51 tells us that

$$\lambda_{((G^v)^w[S \cup \{v\}])^w - w} = \lambda_{G[(S \setminus \{w\}) \cup \{v\}]^v} = \lambda_{G[(S \setminus \{w\}) \cup \{v\}]}.$$

We conclude that

$$\begin{aligned} q(H^v - v) - q(H^v - v - w) &= \sum_{w \in S \subseteq V(H^v - v)} (x - 1)^{|S|} \cdot \lambda_{G[(S \setminus \{w\}) \cup \{v\}]} \left(1 + \frac{y-1}{x-1}\right) \\ &= \sum_{v \in S \subseteq V(G-w)} (x - 1)^{|S|} \cdot \lambda_{G[S]} \left(1 + \frac{y-1}{x-1}\right). \end{aligned}$$

Combining this with equation (2), we see that

$$\begin{aligned} q(G) - q(G - v) &= q(H^v - v) - q(H^v - v - w) + \sum_{v, w \in S \subseteq V(G)} (x - 1)^{|S|} \cdot \lambda_{G[S]} \left(1 + \frac{y-1}{x-1}\right). \end{aligned}$$

Suppose now that  $v, w \in S \subseteq V(G)$ . Then Proposition 50 and Theorem 13 imply that

$$\lambda_G[S] = \lambda_{M_A(G[S])/w} = \lambda_{(G[S]^v)^w - w} = \lambda_{(G^v)^w[S] - w} = \lambda_{H[S \setminus \{w\}]}.$$

As  $v$  is looped in  $H$ , Corollary 51 states that

$$\lambda_{H[S \setminus \{w\}]} = \lambda_{H[S \setminus \{w\}]^v - v} = \lambda_{H^v[S \setminus \{v, w\}]}.$$

We conclude that

$$\begin{aligned} q(G) - q(G - v) &= q(H^v - v) - q(H^v - v - w) + \sum_{v, w \in S \subseteq V(G)} (x-1)^{|S|} \cdot \lambda_{G[S]} \left(1 + \frac{y-1}{x-1}\right) \\ &= q(H^v - v) - q(H^v - v - w) + \sum_{v, w \in S \subseteq V(G)} (x-1)^{|S|} \cdot \lambda_{H^v[S \setminus \{v, w\}]} \left(1 + \frac{y-1}{x-1}\right) \\ &= q(H^v - v) - q(H^v - v - w) + \sum_{S \subseteq V(H^v - v - w)} (x-1)^{|S|+2} \cdot \lambda_{H^v[S]} \left(1 + \frac{y-1}{x-1}\right) \\ &= q(H^v - v) - q(H^v - v - w) + (x-1)^2 \cdot q(H^v - v - w). \end{aligned}$$

This yields the second formula of the recursive description of  $q$ .

In the years since Arratia, Bollobás and Sorkin introduced the interlace polynomials [2, 3, 4], several related graph polynomials have been studied by other researchers [1, 14, 26]. These related polynomials have definitions similar to Definition 46, as sums involving  $GF(2)$ -nullities of symmetric matrices. Consequently they have similar connections with the leading terms of Tutte polynomials of adjacency matroids.

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